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ON THE RESPONSE OF CONTINUOUS MEDIA TO RANDOM EXCITATIONS

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Abstract—This paper is concerned with the formulation of stochastic boundary value problems applied to a linearly elastic continuum subjected to a statistically correlated, multiple random excitation vector field. This formulation involves the description of the correlation tensor and the spectral density tensor of the field of the response of the solid in terms of the respective tensors of the excitation field.

1. INTRODUCTION

THE BEHAVIOR of a finite-number-of-degrees-of-freedom, linear dynamical system subjected to a random excitation has been fully treated by many authors during the last few decades [1–5]; and a great number of physically important problems have been solved [6–8]. Little effort has been devoted, however, to the study and formulation of the continuum problem. Most of the works in this area consider special problems [9–12]. F. P. Beer was the first to attempt formulating the response of a linear mechanical system to a multiple, random, scalar field [13]. By making an extensive use of the multidimensional harmonic analysis, Beer developed many results which may be applied to a very special class of problems.

In this study, the problem of a linear continuum subjected to a random forcing field is considered. The notion of the unit impulse response function and its Fourier transform (the transfer function) used in communication theory [4] is extended to include the respective tensorial quantities. This extension will permit us to study the statistical properties of a linearily elastic solid subjected to a random vector field. We shall extend the formalism of the deterministic boundary value problems of the theory of elasticity to the stochastic boundary value problems and outline some methods of solutions.

2. STATISTICAL SPECIFICATION OF A RANDOM VECTOR FIELD

Consider a volume V bounded by a regular [14] surface Σ. We identify the points of this volume by a position vector $\vec{r}$ referred to a fixed origin o. This is illustrated in Fig. 1 and we select a generic point P of this volume and define a vector $\vec{F} = \vec{F}(\vec{r}, t)$ at this point. The totality of vectors $\vec{F}$ defined for all points of V shall be referred to as a vector field. In the following, we shall work with the components of this vector field referred to a general curvilinear coordinate system in a three-dimensional Euclidian space with the metric tensor $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$; $i, j = 1, 2, 3$ [15].

Let us assume that we have a large number of these volumes V, each with a boundary Σ. Let us also assume that for each of these volumes a vector field $\vec{F}$ is arbitrarily defined. Therefore, we will consider an ensemble of these vector fields, say $\overset{(k)}{\vec{F}}(\vec{r}, t)$; $k = 1, 2, \ldots,$

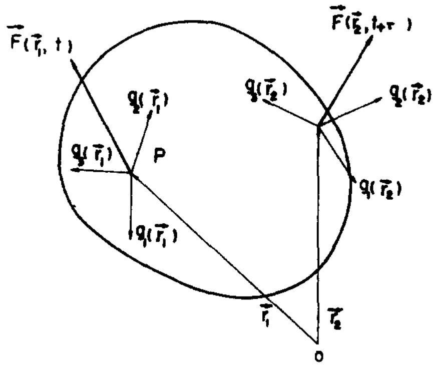

FIG. 1.

$n, \ldots,$ defined, respectively, in the volumes $V_{(k)}$; $k = 1, 2, \ldots, n, \ldots$. In defining our probability density functions, we consider at any time t_1 and at a point $\vec{r}_1$ in V, the fraction of $\overset{(k)}{\vec{F}}(\vec{r}, t)$ whose jth covariant component has, at that instant, a value ranging between given scalar quantities y_1 and $y_1 + \Delta y_1$. For small values of Δy_1, this fraction is proportional to Δy_1 and may be denoted by $P_1(f_j, \vec{r}_1, t_1)\Delta y_1$. We call $P_1(f_j, \vec{r}_1, t_1)$ the first probability density function of this ensemble of vector fields $\overset{(k)}{\vec{F}}(\vec{r}, t) = \overset{(k)}{f_j}(\vec{r}, t)\, \mathbf{g}^j(\vec{r})$. In a similar manner, we consider that fraction of $\overset{(k)}{\vec{F}}(\vec{r}, t)$ whose jth covariant component at time t_1 and at point $\vec{r}_1$ has a value ranging between given scalar quantities y_1 and $y_1 + \Delta y_1$, and whose ith covariant component at time t_2 at point $\vec{r}_2$ has a value ranging between y_2 and $y_2 + \Delta y_2$. For small values of Δy_1 and Δy_2 this fraction is proportional to $\Delta y_1 \Delta y_2$ and may be denoted by $P_2(f_j, f_i, \vec{r}_1, \vec{r}_2, t_1, t_2)\, \Delta y_1\, \Delta y_2$. We call P_2 the second probability density function. This process may be continued and third, fourth, and all higher probability density functions of the ensemble may subsequently be defined. If our knowledge of a vector field $\vec{F}(\vec{r}, t)$ is in a probabilistic sense, as described above, then we shall define this field as a random vector field.

The study of a random vector field in the above sense is, of course, quite complicated. One must consider a great number of vector fields. Furthermore, even if such an ensemble is available, the mathematical formulation of the system subject to no restrictive assumptions is very difficult. For these reasons, we usually consider those vector fields which satisfy certain restrictions. We shall now consider these, beginning with the stationary random vector field. We define a random vector field $\vec{F}(\vec{r}, t)$ to be stationary if the nth probability density function of this vector field is invariant under an arbitrary translation in time; i.e.

$$\begin{aligned} &P_n(f_{j_1}, f_{j_2}, \ldots, f_{j_n}, \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n, t_1, t_2, \ldots, t_n) \\ &= P_n(f_{j_1}, f_{j_2}, \ldots, f_{j_n}, \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n, t_1 + \tau, t_2 + \tau, \ldots, t_n + \tau). \end{aligned} \tag{2.1}$$

Therefore, a stationary vector field may be defined by the following probability functions:

$P_1(f_j, \vec{r}_1)\, \mathrm{d}y_1$, the probability of having the jth covariant component of $\vec{F}$ between the values y_1 and $y_1 + \mathrm{d}y_1$ at point $\vec{r}_1$ and at any time. $P_2(f_{j_1}, f_{j_2}, \vec{r}_1, \vec{r}_2, \tau)\, \mathrm{d}y_1\, \mathrm{d}y_2$, the probability of having the j_1th covariant component of $\vec{F}$ at point $\vec{r}_1$ at time t_1 between the values y_1 and $y_1 + dy_1$, and simultaneously having the j_2th covariant component of $\vec{F}$ at point $\vec{r}_2$ at time $t_1+\tau$ between the value y_2 and y_2+dy_2 for all values of t_1.

$P_3(f_{j_1}, f_{j_2}, f_{j_3}, \vec{r}_1, \vec{r}_2, \vec{r}_3, \tau_1, \tau_2)\,dy_1\,dy_2\,dy_3$, the probability of having $f_{j_1}(\vec{r}_1, t)$, $f_{j_2}(\vec{r}_2, t+\tau_1)$, and $f_{j_3}(\vec{r}_3, t+\tau_1+\tau_2)$, respectively, between the values (y_1, y_1+dy_1), (y_2, y_2+dy_2), and (y_3, y_3+dy_3) for any choice of t.

In short, the stationary random vector field is a vector field for which the ensemble averages are invariant under an arbitrary translation along the time axis. For example, we may define the second correlation tensor of a stationary random vector field by (see Fig. 1).

$$R_{ij}(\vec{r}_1, \vec{r}_2, \tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_i(\vec{r}_1, t) f_j(\vec{r}_2, t+\tau) P_2(f_i, f_j, \vec{r}_1, \vec{r}_2, \tau)\,dy_1\,dy_2. \tag{2.2}$$

There is a subclass of the above random process for which the time averages may be substituted for the ensemble averages.* A random vector field which satisfies this latter assumption is called an ergodic process. For example, for an ergodic random vector field equation (2.2) may be reduced to

$$R_{ij}(\vec{r}_1, \vec{r}_2, \tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_i(\vec{r}_1, t)\, f_j(\vec{r}_2, t+\tau)\,dt. \tag{2.3}$$

Similar results may be obtained for higher order correlation tensors. In general, the components of nth order correlation tensor of an ergodic random vector field may be defined as

$$R_{j_1 j_2 \cdots j_n}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n, \tau_1, \tau_2, \ldots, \tau_{n-1}) = \lim_{T\to\infty}\frac{1}{2T}$$

$$\int_{-T}^{T} f_{j_1}(\vec{r}_1, t) f_{j_2}(\vec{r}_2, t+\tau_1)\ldots f_{j_n}(\vec{r}_n, t+\tau_{n-1})\,dt,$$

$$\tau_1 < \ldots < \tau_{n-1}, j_1, j_2, \ldots, j_n = 1, 2, 3. \tag{2.4}$$

Note that this is the component of an n point tensor. The complete tensor may be written as

$$\begin{aligned} &R(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n, \tau_1, \tau_2, \ldots \tau_{n-1}) \\ &= R_{j_1, j_2 \ldots j_n}(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n, \tau_1, \tau_2, \ldots, \tau_{n-1})\, \mathbf{g}^{j_1}(\vec{r}_1)\mathbf{g}^{j_2}(\vec{r}_2)\ldots \mathbf{g}^{j_n}(\vec{r}_n). \end{aligned} \tag{2.5}$$

We also note that, for a random vector field whose fluctuations are caused by a large number of independent sources, we may approximate the probability density function of the field by a normal density function. Therefore, for such a field with zero mean values, the nth order joint probability density function may be approximated by a Gaussian probability density function defined as

$$\begin{aligned} &P_n(f_{j_1}, f_{j_2}, \ldots, {}_{j_n}, \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n, \tau_1, \tau_2, \ldots, \tau_{n-1}) \\ &= \frac{1}{(2\pi)^{n/2}\|R\|^{1/2}} \exp\left\{-\frac{1}{2}\sum_{i,k=1}^{n} \frac{f_{j_i} f_{j_k} R_{ik}}{\|R\|}\right\} \end{aligned}$$

* The conditions which assure the existence of these time averages are that $\vec{F}$ be finite and continuous in mean-square sense with respect to its time argument. See equation (2.3).

where:

$$f_{j_i} = f_{j_i}(\vec{r}_i, t+\tau_{i-1}),\ \|R\| = \det(R_{ik}),$$
$$R_{ik} = E[f_{j_i} f_{j_k}],\ j_i = 1, 2, 3, \qquad i, k = 1, 2, \ldots, n \tag{2.6}$$

and $E[\quad]$ denotes the ensemble average. We see that for a Gaussian random vector field all order probability density functions are defined when we know the second order correlation tensor of the field.

We now consider another class of random vector fields; the homogeneous, stationary, random, vector field [16]. A random vector field is said to be homogeneous if the nth order correlation tensor of the field is a function of relative position vectors of the considered n points in space. Consider n points of volume V and identify them, respectively, by

$$\vec{r}, \vec{r}+\vec{\rho}_1, \vec{r}+\vec{\rho}_2, \ldots, \vec{r}+\vec{\rho}_{n-1}.$$

The components of the nth order correlation tensor of a homogeneous, stationary, random, vector field are

$$R_{j_1 j_2 \ldots j_n}(\vec{\rho}_1, \vec{\rho}_2, \ldots, \vec{\rho}_{n-1}, \tau_1, \tau_2, \ldots, \tau_{n-1})$$
$$= E[f_{j_1}(\vec{r}, t) f_{j_2}(\vec{r}+\vec{\rho}_1, t+\tau_1) \ldots f_{j_n}(\vec{r}+\vec{\rho}_{n-1}, t + \tau_{n-1})],$$
$$j_1, j_2, \ldots, j_n = 1, 2, 3, \tag{2.7}$$

which indicates that the statistical properties of the field are invariant under an arbitrary translation in space and in time.

Although we have been specifying a random vector field by a complete description of all probability density functions, in practice, it may be considered to be sufficient to give only the second order correlation tensor of the field. This would be justifiable especially if the sources which contribute to the fluctuation of the field are great in number and independent of each other. We shall restrict ourselves to the second order correlation tensor description and assume that it is a satisfactory specification of the random vector field in question.

As we shall see later, it will prove useful to take the Fourier transform of the correlation tensor with respect to time and obtain the spectral density tensor of the random vector field. If a function $f(t)$ is absolutely integrable, that is, if $\int_{-\infty}^{\infty} |f(t)|\, \mathrm{d}t < \infty$, then it can be represented in the Fourier integral form [18],

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(i\omega)\, \mathrm{e}^{i\omega t}\, \mathrm{d}\omega$$

where

$$F(i\omega) = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-i\omega t}\, \mathrm{d}t.$$

We also have Parseval's equality for $f(t)$ of finite norm:

$$\|F(i\omega)\|^2 = 2\pi \|f(t)\|^2$$

where $\|F\|$ denotes the norm of F.

Consider a representative member of an ensemble of vector fields which satisfy the condition of absolute integrability (with respect to time for all points in V and on Σ). The Fourier integral representation of this field is

$$f_j(\vec{s}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_j(\vec{s}, \omega)\, e^{i\omega t}\, d\omega$$

where

$$F_j(\vec{s}, \omega) = \int_{-\infty}^{\infty} f_j(\vec{s}, t)\, e^{-i\omega t}\, dt; \quad j = 1, 2, 3.$$

Let us now define the power spectral density tensor of an ensemble of a stationary vector field by

$$\overset{F}{\phi}_{ij}(\vec{s}, \vec{\rho}; \omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} f_i(\vec{s}, t)\, e^{i\omega t}\, dt \int_{-T}^{T} f_j(\vec{s} + \vec{\rho}, t')\, e^{-i\omega t'}\, dt'\right] \tag{2.8}$$

where the limit in (2.8) must exist independently of the particular choice of T. In this case, by changing the order of summation and integration, we get

$$\overset{F}{\phi}_{ij}(\vec{s}, \vec{\rho}; \omega) = \int_{-\infty}^{\infty} \overset{F}{R}_{ij}(\vec{s}, \vec{\rho}, \tau)\, e^{-i\omega\tau}\, d\tau; \quad t' - t = \tau, \qquad i, j = 1, 2, 3. \tag{2.9}$$

Similarly, for an ergodic, stationary random field we define

$$\overset{F}{\phi}_{ij}(\vec{\rho}; \omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-i\omega\tau}\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_i(\vec{s}, t) f_j(\vec{s} + \vec{\rho}, t + \tau)\, dt\right\} d\tau$$

$$= \int_{-\infty}^{\infty} \overset{F}{R}_{ij}(\vec{\rho}; \tau)\, e^{-i\omega\tau}\, d\tau. \tag{2.10}$$

3. DESCRIPTION OF A STOCHASTIC, LINEARLY ELASTIC, BOUNDARY VALUE PROBLEM:

Consider a linearly elastic solid with volume V and regular boundary Σ. We identify the points of this solid by a position vector $\vec{r}$ referred to a fixed point 0 and study the statistical parameters of its response to a random excitation applied on its boundary Σ. The excitation may be random tractions, random displacements or some combination of both. In this way, we may have three distinct stochastic boundary value problems. Consider, for example, the first boundary value problem and assume that we are interested in formulating the statistical parameters of the displacement field of this linearly elastic solid in terms of the statistical parameters of the applied random tractions. When we speak of the traction field being statistically defined, we mean that at a point on Σ and at time t we are able to give the probability of having the intensity of the components of the tractions within a given range. If we had available all the members of the ensemble of the input, it would be possible (at least theoretically) to determine the response for each member, which response, for the particular boundary conditions, would satisfy the field equations of linear elasticity. In fact, such input information is not generally available and if it were, then the subsequent analysis would be mathematically prohibitive. It is necessary, therefore, to develop another approach for the formulation of the boundary value problem. To do this, we shall employ the idea of unit impulse response used in solving a finite-number-of-degrees-of-freedom-system and generalize it for application to our boundary value problem.

The response, in our case, may be the covariant components of displacement field, or stress field. To fix attention we define $h_{ij}(\vec{r}, \vec{s}; t_{\vec{r}} - t_{\vec{s}})\,\mathbf{g}^i(\vec{r})\,\mathbf{g}^j(\vec{s})$ to be the displacement tensor at time $t_{\vec{r}}$ at point $\vec{r}$ due to a unit covariant impulse applied at point $\vec{s}$ at time $t_{\vec{s}}$ and $h_{ijk}(\vec{r}, \vec{s}; t_{\vec{r}} - t_{\vec{s}})\,\mathbf{g}^i(\vec{r})\,\mathbf{g}^j(\vec{r})\,\mathbf{g}^k(\vec{s})$ to be the stress tensor at time $t_{\vec{r}}$ at point $\vec{r}$ due to a unit covariant impulse applied at point $\vec{s}$ at time $t_{\vec{s}}$. The impulse is a concentrated impulsive force for the first boundary value problem and a concentrated impulsive displacement for the second boundary value problem. By unit covariant impulse we mean an impulse $\vec{v}(\vec{s}, t_{\vec{s}}) = v^i(\vec{s}, t_{\vec{s}})\,\mathbf{g}_i(\vec{s})$ where $v^1, v^2, v^3 = \delta(\vec{r} - \vec{s})\,\delta(t - t_s)$. We may likewise define contravariant components and mixed components of the unit impulse response. For example, we may define $h_{i\cdot}^{\cdot j}(\vec{r}, \vec{s}; t_{\vec{r}} - t_{\vec{s}})\,\mathbf{g}^i(\vec{r})\,\mathbf{g}_j(\vec{s})$ to be the displacement vector at point $\vec{r}$ at time $t_{\vec{r}}$ due to unit contravariant impulse applied at point $\vec{s}$ at time $t_{\vec{s}}$; where the unit contravariant impulse is defined by $\vec{v}(\vec{s}, t_{\vec{s}}) = v_i(\vec{s}, t_{\vec{s}})\,\mathbf{g}^i(\vec{s})$; $v_1(\vec{s}, t_{\vec{s}})$, $v_2(\vec{s}, t_{\vec{s}})$, $v_3(\vec{s}, t_{\vec{s}}) = \delta(\vec{r} - \vec{s})\,\delta(t - t_{\vec{s}})$.

With the above definitions, the unit impulse response is a tensor with respect to all of its indices. It is obvious that it is a two point tensor referred to a fixed curvilinear coordinate system. For instance, we may immediately conclude the following relations

$$
\begin{aligned}
h_{ij} &= h_{i\cdot}^{\cdot k}\, g_{kj}(\vec{s}) \\
h_{ijk} &= h^{rpq}_{\cdot\cdot\cdot}\, g_{ri}(\vec{r})\, g_{pj}(\vec{r})\, g_{qk}(\vec{s}).
\end{aligned}
\tag{3.1}
$$

If, instead of a general curvilinear coordinate system, we employ a system of orthogonal cartesian coordinates x_j; $j = 1, 2, 3$, then $\mathbf{g}_i$ becomes the unit base vector $\hat{\imath}_i$ and g_{ij} reduces to the Kronecker delta, δ_{ij}. The unit impulse response tensor h_{ij}, in this case, is the displacement measured in the $\hat{\imath}_i$ direction at point $\vec{r}$ at time $t_{\vec{r}}$ due to a unit impulse applied in the $\hat{\imath}_j$ direction at points $\vec{s}$ at time $t_{\vec{s}}$. It is clear that, in this case, h_{ij} is the physical component of the unit impulse response and, consequently, the reciprocity relationship is valid; i.e. $h_{ij}(\vec{r}, \vec{s}; t_{\vec{r}} - t_{\vec{s}}) = h_{ji}(\vec{s}, \vec{r}; t_{\vec{s}} - t_{\vec{r}})$.

Now, let us formalize the three boundary value problems and describe a method of solution.

Problem 1: Determine the correlation tensor of stresses and the correlation tensor of displacements, each of the nth order, in the interior of a linearly elastic solid when the nth order correlation tensor of the applied forces on all points of the surface boundary of the solid is specified.

Problem 2: Determine the correlation tensor of stresses and the correlation tensor of displacements, each of the nth order, in the interior of a linearly elastic solid when the nth order correlation tensor of the imposed displacements on all points of the surface boundary of the solid is specified.

Problem 3: Determine the correlation tensor of stresses and the correlation tensor of displacements, each of the nth order, in the interior of a linearly elastic solid when the nth order correlation tensor of the applied tractions on a part of the surface boundary of the solid is known and on the remaining part of the surface boundary of the solid the nth order correlation tensor of the imposed displacements is specified.

To show a general method of solution, we shall consider problem 1 and for $n = 2$, shall outline a procedure for obtaining the correlation tensor of displacements. The stress correlation tensor may then be formulated using the same line of reasoning.

Let $h_{ij}(\vec{r}, \vec{s}; t_{\vec{r}} - t_{\vec{s}})\, \mathbf{g}^i(\vec{r})\, \mathbf{g}^j(\vec{s}); i, j = 1, 2, 3$, be the unit impulse response tensor of the displacement field of the solid for the first boundary value problem. The response of this linearly elastic solid to a deterministic forcing field $\vec{F}(\vec{s}, t)$ defined on Σ may be written as

$$U_i(\vec{r}, t_{\vec{r}}) = \int_{\Sigma} \int_{-\infty}^{t_{\vec{r}}} h_{ij}(\vec{r}, \vec{s}; t_{\vec{r}} - t_{\vec{s}}) f^j(\vec{s}, t_{\vec{s}})\, d\sigma\, dt_{\vec{s}}; \qquad i, j = 1, 2, 3. \tag{3.2}$$

where $d\sigma$ is an elementary area of Σ, and f^j are the contravariant components of the forcing field. The usual summation convention on the indices is implied in equation (3.2). Since h_{ij} is zero for points outside of Σ and for $t_{\vec{s}} > t_{\vec{r}}$, the limits of the integrals in equation (3.2) may be taken from $-\infty$ to $+\infty$ for both the time and the space variations. Hence with $\tau = t_{\vec{r}} - t_{\vec{s}}$, equation (3.2) becomes

$$U_i(\vec{r}, t) = \int\int h_{ij}(\vec{r}, \vec{s}; \tau) f^j(\vec{s}, t - \tau)\, d\sigma\, d\tau \tag{3.3}$$

and these and all subsequent unlimited integrals extend from $-\infty$ to $+\infty$. The correlation tensor of the displacement field is

$$\overset{u}{R}_{ij}(\vec{r}_1, \vec{r}_2; t_1, t_2) = E[U_i(\vec{r}_1, t_1) U_j(\vec{r}_2, t_2')]; \qquad i, j = 1, 2, 3. \tag{3.4}$$

Employing equation (3.3) we obtain

$$\overset{u}{R}_{ij}(\vec{r}_1, \vec{r}_2; t_1, t_2) = E\left[\int\int h_{i\nu}(\vec{r}_1, \vec{s}_1; \tau_1) f^\nu(\vec{s}_1, t_1 - \tau_1)\, d\sigma_1\, d\tau_1 \right. \tag{3.5}$$
$$\left. \int\int h_{j\mu}(\vec{r}_2, \vec{s}_2; \tau_2) f^\mu(\vec{s}_2, t_2 - \tau_2)\, d\sigma_2\, d\tau_2 \right]; \quad i, j, \nu, \mu = 1, 2, 3.$$

For a stationary random input we set $t_2 = t_1 + \tau = t + \tau$ and obtain

$$\overset{u}{R}_{ij}(\vec{r}_1, \vec{r}_2; \tau) = E\left[\int\int h_{i\nu}(\vec{r}_1, \vec{s}_1; \tau_1) f^\nu(\vec{s}_1, t - \tau_1)\, d\sigma_1\, d\tau_1 \right.$$
$$\left. \int\int h_{j\mu}(\vec{r}_2, \vec{s}_2; \tau_2) f^\mu(\vec{s}_2, t + \tau - \tau_2)\, d\sigma_2\, d\tau_2 \right] \tag{3.6}$$

If we set

$$\vec{r}_1 = \vec{r}, \vec{r}_2 = \vec{r}_1 + \vec{\rho} = \vec{r} + \vec{\rho}, \tau + \tau_1 - \tau_2 = \tau', \vec{s}_2 = \vec{s}_1 + \vec{\rho}_1,$$

we obtain

$$\overset{u}{R}_{ij}(\vec{r}, \vec{\rho}; \tau) = \int\int\int\int h_{i\nu}(\vec{r}, \vec{s}_1; \tau_1) h_{j\mu}(\vec{r} + \vec{\rho}, \vec{s}_1 + \vec{\rho}_1; \tau_2)$$
$$\overset{F}{R}{}^{\nu\mu}(\vec{s}_1, \vec{\rho}_1; \tau')\, d\sigma_1, d\sigma_2\, d\tau_1 d\tau_2, \tag{3.7}$$

and for a homogeneous, stationary input this reduces to

$$\overset{u}{R}_{ij}(\vec{r},\vec{\rho};\tau) = \iiiint h_{i\nu}(\vec{r},\ \vec{s}_1;\tau_1)h_{j\mu}(\vec{r}+\vec{\rho},\vec{s}_1+\vec{\rho}_1;\tau_2)\,\overset{F}{R}^{\nu\mu}(\vec{\rho}_1;\tau')\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2. \tag{3.8}$$

From equation (3.8) it is clear that the response of a linearly elastic system to a homogeneous, stationary, random loading, in general, is a non-homogeneous, stationary, random field.

Instead of working in the time domain we may formulate the problem in the frequency domain by using the Fourier transform technique. Let

$$\begin{aligned} H_{ij}(\vec{r},\vec{s};\omega) &= \int_{-\infty}^{\infty} h_{ij}(\vec{r},\vec{s};\tau)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\tau \\ h_{ij}(\vec{r},\vec{s};\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{ij}(\vec{r},\vec{s};\omega)\,\mathrm{e}^{i\omega\tau}\,\mathrm{d}\omega \end{aligned} \tag{3.9}$$

$$\begin{aligned} \overset{u}{\phi}_{ij}(\vec{r}_1,\vec{r}_2;\omega) &= \int_{-\infty}^{\infty} \overset{u}{R}_{ij}(\vec{r}_1,\vec{r}_2;\tau)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\tau \\ \overset{u}{R}_{ij}(\vec{r}_1,\vec{r}_2;\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overset{u}{\phi}_{ij}(\vec{r}_1,\vec{r}_2;\omega)\,\mathrm{e}^{i\omega\tau}\,\mathrm{d}\omega. \end{aligned} \tag{3.10}$$

where H_{ij} and ϕ_{ij} are complex tensors such that†

$$\int_{-\infty}^{\infty} |H_{ij}|\mathrm{d}\omega < \infty \text{ and } \int_{-\infty}^{\infty} |\phi_{ij}|\mathrm{d}\omega < \infty$$

for all points interior to and on the boundary of the solid. Equations (3.10) are valid for a stationary process only. Taking the Fourier transform of equation (3.7) we write

$$\phi_{ij}(\vec{r},\vec{\rho};\omega) = \iint\left\{\lim_{t\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T}\int_{-T}^{T} h_{i\nu}(\vec{r},\vec{s}_1;\tau_1)h_{j\mu}(\vec{r}+\vec{\rho},\vec{s}_1+\vec{\rho}_1;\tau_2)\,\overset{F}{R}^{\nu\mu}(\vec{s}_1,\vec{\rho}_1;\tau+\tau_1-\tau_2)\,\mathrm{e}^{-i\omega\tau}\,\mathrm{d}\tau\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\right\}\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2.$$

The limits must exist independently of a particular choice of T. Using this condition we obtain

$$\overset{u}{\phi}_{ij}(\vec{r},\vec{\rho};\omega) = \iint H^{*}_{i\nu}(\vec{r},\vec{s}_1;\omega)H_{j\mu}(\vec{r}+\vec{\rho},\vec{s}_1+\vec{\rho}_1;\omega)\,\overset{F}{\phi}^{\nu\mu}(\vec{s}_1,\vec{\rho}_1;\omega)\,\mathrm{d}\sigma_1\,\mathrm{d}\sigma_2, \tag{3.11}$$

where H^{*}_{ij} is the complex conjugate of H_{ij}.

† H_{ij} will be called the transfer tensor of the system.

It is often convenient to introduce the concept of generalized Fourier transform as follows

$$H_{ij}(\vec{r}, \vec{k}; \omega) = \iint h_{ij}(\vec{r}, \vec{s}; \tau)\, e^{-i(\vec{k}\cdot\vec{s}+\omega\tau)}\, d\sigma\, d\tau$$
$$h_{ij}(\vec{r}, \vec{s}; \tau) = \left(\frac{1}{2\pi}\right)^4 \iint H_{ij}(\vec{r}, \vec{k}; \omega)\, e^{i(\vec{k}\cdot\vec{s}+\omega\tau)}\, d\vec{k}\, d\omega. \tag{3.12}$$

where $\vec{k}$ is wave vector. The integrals in (3.12) are assumed to exist. This condition is satisfied when $\int\int |H_{ij}|\, d\vec{k}\, d\omega < \infty$ for all points inside and on Σ. Physically, $H_{ij}(\vec{r}, \vec{k}; \omega)$ $e^{i\omega t}$ are the components of the response of a solid to generalized harmonic tractions of the form

$$f^1(\vec{s}, t), f^2(\vec{s}, t), f^3(\vec{s}, t) = e^{-i(\vec{k}\cdot\vec{s}+\omega t)};\ i = (\sqrt{-1}). \tag{3.13}$$

With the use of the above concept, equation (3.11) reduces to a simple form in certain special cases. For instance, when the unit impulse response tensor of the solid is only dependent upon the relative position of the points $\vec{r}$ and $\vec{s}$, $h_{ij}(\vec{r}, \vec{s}; t_{\vec{r}}-t_{\vec{s}}) = h_{ij}(\vec{r}-\vec{s}; t_{\vec{r}}-t_{\vec{s}}) = h_{ij}(\vec{\rho}; \tau)$†. In this case, equation (3.8) reduces to

$$\overset{u}{R}_{ij}(\vec{\rho}; \tau) = \iiiint h_{i\nu}(\vec{s}_1; \tau_1) h_{j\mu}(\vec{s}_2; \tau_2) \overset{F}{R}{}^{\nu\mu}(\rho'; \tau')\, d\sigma_1\, d\sigma_2\, d\tau_1\, d\tau_2$$

where

$$\rho' = \vec{\rho}+\vec{s}_1-\vec{s}_2,\ \tau' = \tau+\tau_1-\tau_2. \tag{3.8'}$$

Taking the generalized Fourier transform of (3.8′) we obtain

$$\overset{u}{\phi}_{ij}(\vec{k};\omega) = H^*_{i\nu}(\vec{k}; \omega) H_{j\mu}(\vec{k}; \omega) \phi^{\nu\mu}(\vec{k}; \omega); \qquad i, j, \nu, \mu = 1, 2, 3, \tag{3.14}$$

where $H^*_{ij}(\vec{k}; \omega)$ is complex conjugate of $H_{ij}(\vec{k}; \omega)$ in a generalized sense. We note that equation (3.14) is valid only when the forcing field is a homogeneous, stationary, random vector field, and the components of the unit impulse response tensor of the solid are functions of the relative positions of the point of the application of the unit impulse and the point at which the response is measured. This latter requirement not only restricts the geometry of the solid to which equation (3.14) is applicable, but also limits the spatial configuration of the region in which the random excitation is exerted. In certain problems, by limiting the geometry of the region in which a homogeneous, stationary, random vector field may be defined, we may be able to employ equation (3.14). The following lists some interesting problems which may be solved using equation (3.14) and the usual equations of linear elasticity.

1. An infinite solid subject to a homogeneous, stationary, random vector field which is distributed in a region within the solid.
2. A half space subjected to a homogeneous, stationary, random, vector field on its boundary.
3. An infinite plate subjected to a homogeneous, stationary, random, vector field.

† For example the cases of infinite plate, infinite beam, and infinite string. We note that in [13] this important limitation is not realized.

4. A semi-infinite plate shown in (Fig. 2), which has the same edge conditions at all points on I and II. Equation (3.14) may be used only when the homogeneous, stationary, random, pressure field is applied on $yy-$axis.

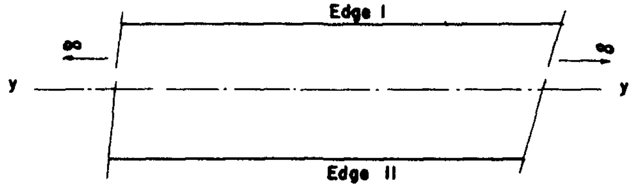

FIG. 2.

5. An infinite beam or string subjected to a homogeneous, stationary, random, scalar field.

In general, from equation (3.7) we conclude that, for a linearly elastic finite solid the nonhomogeneity of the response is independent of whether the forcing field is homogeneous or not.

We summarize these facts with the following conclusion:

The response of a linearly elastic solid to a homogeneous or a nonhomogeneous, stationary, random excitation is, in general, a nonhomogeneous, stationary, random field. In a very special case, when the unit impulse response tensor of the solid is a function of the relative position of the point of the solid at which the impulse is applied and the point of the solid at which the response is sensed, then the response of the solid to only a homogeneous, stationary, random excitation is a homogeneous, stationary, random field.

Before closing this section, let us outline a method of finding the unit impulse response for the first boundary value problem in rectangular cartesian coordinate system. The equations of equilibrium and the boundary conditions are

$$\frac{\partial \sigma_{ij}}{\partial x_j} - \rho \frac{\partial^2 U_i}{\partial t^2} = 0, \qquad \text{in } V$$

$$\sigma_{ij} = \lambda \delta_{ij} U_{k,k} + \mu (U_{i,j} + U_{j,i})$$

$$\sigma_{ij} \nu_j = p_i(x_1, x_2, x_3, t), \qquad \text{on } \Sigma \tag{3.15}$$

Let the eigenvalues of the system (3.15) (with $p_i = 0$ and $U_i = \psi_{in} e^{i\omega t}$) be $\omega_1 \leqslant \omega_2 \leqslant \ldots \leqslant \infty$. To these eigenvalues there correspond eigenvectors $\psi_{i1}(x_1, x_2, x_3)$, $\psi_{i2}(x_1, x_2, x_3), \ldots$ which form a complete vector space satisfying the orthogonality condition

$$\int_V \rho \psi_{im} \psi_{in} \, dv = \begin{cases} 0 & , m \neq n \\ \|\psi_n\|^2, & m = n; \end{cases} \|\psi_n\|^2 = \int_V \rho \psi_{in} \psi_{in} \, dv.$$

To find the transfer function of the system we let $p_1, p_2, p_3 = \delta(x_1 - x_{01})\, \delta(x_2 - x_{02})\, \delta(x_3 - x_{03})\, e^{i\omega t}$ and obtain

$$H_{kj}(x_i, x_{0i}; \omega) = \sum_n \frac{\bar{a}_{jn} \psi_{kn}(x_i)}{[(i\omega)^2 + \omega_n^2]}, \qquad \bar{a}_{jn} = \frac{[\psi_{jn}(x_{01}, x_{02}, x_{03})]}{\|\psi_n\|^2}. \tag{3.16}$$

We may include normal viscous damping of the form $2\rho\eta_1(\partial U_i/\partial t)$ and obtain

$$H_{kj}(x_i, x_{0i}; \omega) = \sum_n \frac{\bar{a}_{jn}\psi_{kn}(x_i)}{[(i\omega)^2 + 2\eta_1(i\omega) + \omega_n^2]}. \tag{3.17}$$

The unit impulse function is obtained by taking the Fourier transform of (3.17)

$$h_{kj}(x_i, x_{0i}; \tau) = \sum_n \frac{e^{-\eta'\omega_n\tau}\sin(\sqrt{1-\eta'^2}\,\omega_n\tau)\bar{a}_{jn}\psi_{kn}(x_i)}{\omega_n\sqrt{1-\eta'^2}} \qquad \tau \geqslant 0,\ \eta' < 1 \tag{3.18}$$

where we have taken η_1 proportional to ω_n; i.e. $\eta_1 = \eta'\omega_n$.

In the following section, we shall formulate the probability density function of the displacement field of a circular cylindrical shell subjected to axisymmetrically applied, purely random, Gaussian loading. Although this is a one-dimensional problem and, therefore, does not illustrate all the results obtained in the previous sections, it leads to physically interesting results which may be worthy of notice.

4. EXAMPLE

We consider a simply supported, circular, cylindrical, thin shell, and first formulate its unit impulse response function when subjected to a ring-loading. With the Kirchhoff hypothesis, the equation of motion is

$$D\frac{\partial^4 w}{\partial x^4} + \frac{Eh}{a^2}w + m\frac{\partial^2 w}{\partial t^2} + 2m\eta_1\frac{\partial w}{\partial t} = p(x, t), \qquad w = \frac{\partial^2 w}{\partial x^2}; \quad x = 0, L,$$

where

L = length of the shell
w = radial displacement
h = thickness of the shell
m = mass density per unit area of the middle surface
$2\eta_1$ = damping per unit of m
$D = Eh^3/12(1-v^2)$
v = Poisson's ratio
E = Young's modulus
$p(x, t)$ = the radially applied forcing function at the section x and the time t.

It may easily be verified that

$$\{\omega_n\} = \sqrt{\frac{D}{m}}\left\{\left[\frac{n^4\pi^4}{L^4} + \frac{Eh}{Da^2}\right]^{\frac{1}{2}}\right\}$$

are the eigenvalues corresponding to the eigenfunctions

$$\{\psi_n\} = \sqrt{\frac{2}{L}}\left\{\sin\frac{n\pi x}{L}\right\},$$

of the free vibrations of this system. The unit impulse function may now be written using equation (3.18),

$$h_{rr}(x, x_0; \tau) = \frac{2D^2}{Lm^2} \sum_n \left\{ \sin\frac{n\pi x_0}{L} \sin\frac{n\pi x}{L} \frac{\sin(\sqrt{1-\eta'^2}\,\omega_n \tau)}{\omega_n \sqrt{1-\eta'^2}} e^{-\eta'\omega_n \tau} \right\}.$$

For a purely random, radial loading, the correlation function is given by

$$\overset{p}{R}_{rr}(x'_0, x''_0; \tau) = R_0\, \delta(x'_0 - x''_0)\, \delta(\tau),$$

where R_0 is a constant. Using equation (3.7), we may immediately arrive at

$$\overset{w}{R}_{rr}(x_1, x_2; \tau) = \frac{R_0}{2L}\sqrt{\frac{D}{m}} \sum_n \left\{ \frac{\sin(n\pi x_1)/L \sin(n\pi x_2)/L}{[(n^4\pi^4/L^4)+(EL/Da)]^{\frac{3}{2}}} \left[\frac{\sin(\sqrt{1-\eta'^2}\,\omega_n\tau)}{\sqrt{1-\eta'^2}} + \frac{\cos(\sqrt{1-\eta'^2}\,\omega_n\tau)}{\eta'} \right] e^{-\eta'\omega_n\tau} \right\}, \tag{*}$$

where

$$\omega_n = \sqrt{\frac{D}{m}} \left[\frac{n^4\pi^4}{L^4} + \frac{Eh}{Da^2} \right]^{\frac{1}{2}}.$$

For $\tau = 0$, we get the cross-variance of the displacements at points x_1 and x_2; i.e.

$$\overset{w}{R}_{rr}(x_1, x_2; 0) = \sqrt{\frac{D}{m}} \left(\frac{R_0}{2L\eta'} \right) \sum_n \left\{ \frac{\sin(n\pi x_1)/L \sin(n\pi x_2)/L}{\{(n^4\pi^4/L^4)+[12(1-\nu^2)/a^2h^2]\}^{\frac{3}{2}}} \right\}$$

which may be made dimensionless by letting $x_1/L = y_1$, $x_2/L = y_2$, $a/L = A$, and $a/h = c$. Then

$$\overset{w}{R}_{rr}(y_1, y_2; 0) = \sqrt{\frac{D}{m}} \left(\frac{R_0 a^6}{2L\eta'} \right) \bar{G}\,(y_1, y_2; A, c, \nu)$$

where

$$\bar{G}(y_1, y_2; A, c, \nu) = \sum_n \left\{ \frac{\sin(n\pi y_1)\sin(n\pi y_2)}{[n^4\pi^4 A^4 + 12(1-\nu^2)c^2]^{\frac{3}{2}}} \right\}. \tag{**}$$

To show the general form of equation (**) and especially the dominant effect of A, we have plotted $\bar{G}$ for $A = 1{\cdot}0$, and $A = 0{\cdot}1$ in Fig. 3 while keeping $c = 100$, and $\nu = 0{\cdot}20$. From these graphs we see that for $A = a/L = 0{\cdot}1$, $\bar{G}$ has a sharp peak at $y_1 = y_2$ and decreases quite rapidly for $y_1 \neq y_2$ and takes on negative values. This indicates that for A small, there is a very small correlation between the displacements of two spatially separated points of the shell. This conclusion may also be deduced from equation (**). From this equation we see that when A approaches zero (very long shell), $\bar{G}$ become proportional to the Dirac delta function of $(y_1 - y_2)$; i.e. $\bar{G} \sim \delta(y_1 - y_2)$ and there is no correlation between the displacements at y_1 and y_2 for $y_1 \neq y_2$. Let us suppose that the forcing field is also a Gaussian process. In this case, the displacement field will also be a Gaussian random field. From equation (*) we conclude that, for small values of A, the cross-variance of the displacements at two spatially separated points of the shell is very small

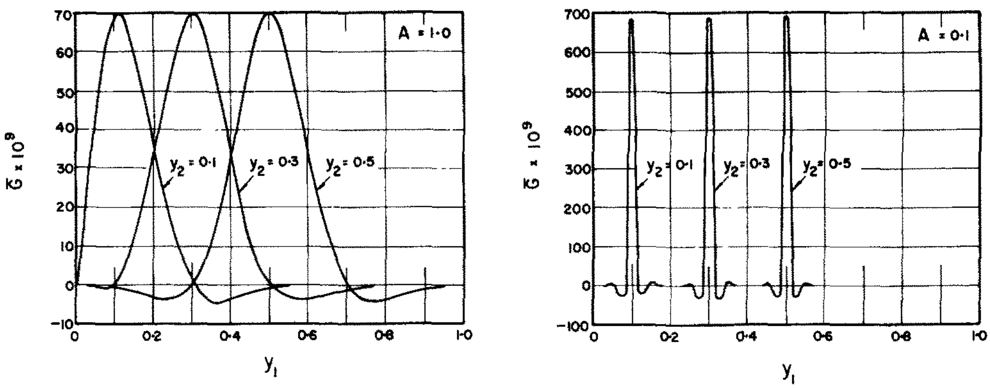

FIG. 3. Cross-variance of the radial displacements of an axisymmetrically loaded circular, cylindrical shell under a purely random input, $c = 100$, $\nu = 0{\cdot}20$.

and may be set equal to zero. Therefore, for $\tau = 0$, the displacement field is completely specified by a first order normal probability density function given by

$$P_1(w;x) = \frac{1}{\sqrt{2\pi R_{rr}(x,x;0)}}\exp\left\{\frac{-w^2}{2R_{rr}(x,x;0)}\right\}.$$

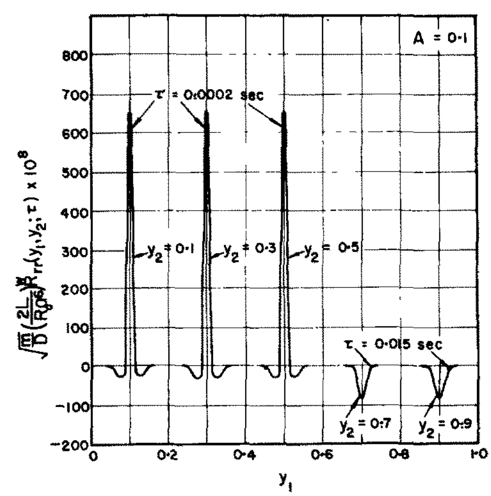

FIG. 4. Cross-correlation function of the radial displacements of an axisymmetrically loaded circular, cylindrical shell under a purely random input, $c = 100$, $h = 2$ in., $\nu = 0{\cdot}2$, $E/\rho = 4{\cdot}1\times10^{10}$ (in/sec)^2, and $\eta' = 0{\cdot}1$.

The second order probability density function then is

$$P_2(w_1w_2;x_1,x_2;0) = P_1(w_1;x_1)\,P_1(w_2;x_2)$$

and similarly, we can write the nth order probability density function of the system. The above results are also valid for the cases of non-zero values of τ. To show this, we simply have to realize that, for fixed values of x_1 and x_2, $\overset{w}{R}_{rr}(x_1, x_2; \tau)$ has its maximum at $\tau = 0$; i.e. $\overset{w}{R}_{rr}(x_1, x_2; 0) \geqslant \overset{w}{R}_{rr}(x_1, x_2; \tau)$. Equation (*) is plotted in Fig. 4 for $A = 0{\cdot}1$, $c = 100$, $h = 2$ in., $\nu = 0{\cdot}2$, $E/\rho = 4.1 \times 10^{10}$ (in/sec)^2, and $\tau = 0{\cdot}0002$ and $\tau = 0{\cdot}015$ sec.

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Résumé—Cette étude s'interesse à la formulation des problèmes de la valeur limite stochastique appliqués à un continuum linéairement élastique sujet à un champ d'excitation vecteur multiple, statisquement en corrélation. Cette formation inclue la description du tenseur de corrélation et le tenseur de densité spectrale du champ de réaction du solide exprimé en tenseurs respectifs du champ d'excitation.

Zusammenfassung—Diese Abhandlung betrifft die Formulierung von stochastischen Randwert Problemen welche zu einem linearen elastischen Kontinuum angewendet werden, die einem statistischen korrelierten, mehrfachen Zufalls Erregungs Vektorfeld unterworfen sind. Diese Formulierung umfasst die Beschreibung des Korrelations Tensors und des spektralen Dichtigkeits Tensors der Feldbeanspruchung des Festkörpers in Ahangigkeit von den respektiven Tensoren des Erregungsfeldes.

Абстракт—Эта статья касается формулировки вероятных (стохастических) краеьых задаг, применяемых к линейно упругому континууму, подвергнутому статистики связанному, многократно произвольному векторному возбуждения магнитного поля. Эта формулировка включает описание тензора соотношения и тензора спектральной плотности поля реакции твёрдых тел через соответствующих тензоров поля возбуждения.